ALGEBRA MAST30005 SEMESTER 1 2017

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Contents

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These are notes for Nora Ganter's MAST30005 Algebra course, written by Jon Xu, tutor for the Thursday 3:15PM practise class. These were written by me for the purpose of processing certain parts of the course, and is not intended to completely cover the material. In fact quite a lot is missing, and some statements are made without proof (exercise: fill in the proofs!). This is DEFINITELY NOT a substitute for attending the lectures/consultations/tutorials.

Use at your own risk! Any constructive comments are welcome, my email address is

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1. Semi-direct products

An *exact sequence* is

$$G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} G_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} G_n,$$

where

- The G_i's are groups,
- The d_i's are group homomorphisms,
- $\ker(d_{i+1}) = \mathrm{im}(d_i)$ for all i.

A *short exact sequence* is an exact sequence

$$1 \to N \xrightarrow{i} G \xrightarrow{p} H \to 1.$$

The above short exact sequence *splits* if there exists a homomorphism $s\colon H \to G$ such that $p \circ s = 1_H$. In this case we write

$$1 \to N \xrightarrow{i} G \underset{s}{\overset{p}{\rightleftarrows}} H \to 1.$$

In this case, we have an action of H on N

$$\begin{array}{rcl} H & \to & \mathrm{Aut}(N) \\ h & \mapsto & s(h)(-)s(h)^{-1} = (-)^h \end{array}$$

Given two groups N and H and an action of H on N, we define the *semidirect product* of N and H to be the set

$$N \rtimes H = \{(n, h) \mid n \in N, h \in H\}$$

with multiplication defined by

$$(n, h)(m, k) = (n \cdot m^h, hk).$$

Lemma 1.1. *If*

$$1 \to N \xrightarrow{i} G \underset{s}{\overset{p}{\rightleftarrows}} H \to 1.$$

is a split short exact sequence, then the map

$$\begin{array}{rcl} \Phi\colon N \rtimes H & \longrightarrow & G \\ (n, h) & \longmapsto & i(n)s(h). \end{array}$$

is an isomorphism of groups.

Proof. Homomorphism.

$$\begin{aligned}\Phi((n,h))\Phi((m,k)) &= i(n)s(h)i(m)s(k) \\ &= i(n)s(h)i(m)s(h)^{-1}s(h)s(k) \\ &= i(n)m^h s(hk) \\ &= \Phi((nm^h, hk)) \\ &= \Phi((n,h)(m,k)).\end{aligned}$$

Bijective. Decompose G into right cosets of N, i.e

$$G = \sqcup_{h\in H} Ns(h)$$

and $\{s(h) \mid h \in H\}$ is a set of representatives. □

Proposition 1.2. *The converse to the above is true (TO DO?).*

Example 1. The following is a split short exact sequence:

$$1 \to SO(2) \xrightarrow{i} O(2) \underset{s}{\overset{p}{\rightleftarrows}} \{\pm 1\} \to 1.$$

where

$$s(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad s(-1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence $O(2) = SO(2) \rtimes \{\pm 1\}$.

The *Klein four group* is $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The *tetrahedral group* T is the subgroup of $SO(3)$ that stabilises a tetrahedron centered at 0 in $\mathbb{R}^3$. That is,

$$T = \{g \in SO(3) \mid g \cdot A = A\},$$

where A is an tetrahedron with center at 0.

The *alternating group* A_n is the subgroup of S_n of permutations that can be written as a product of an even number of transpositions. Note that $T \cong A_4$.

The *octohedral group* O is the subgroup of $SO(3)$ that stabilises an octohedron centered at 0 in $\mathbb{R}^3$. Note that $O \cong S_4$.

The *icoshedral group* I is the subgroup of $SO(3)$ that stabilises an icosohedron centered at 0 in $\mathbb{R}^3$. Note that $I \cong S_5$.

Proposition 1.3. *Let* $A \in SO(3)$. *There exists positively oriented orthonormal basis* $\mathcal{B} = \{b_1, b_2, b_3\}$ *such that*

$$[A]_{\mathcal{B}} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In other words, there exists an invertible matrix B *(WITH POSITIVE DETERMINANT?) such that*

$$A = B \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} B^{-1}.$$

Theorem 1.4 (Artin, Theorem 6.12.1)**.** *Let* H *be a finite subgroup of* $SO(3)$. *Then exactly one of the following is true:*

- *H is a cyclic group C_k of rotations by mutiples of $2\pi/k$ about a line, with $k \in \mathbb{Z}_{\geq 1}$,*
- *H is a dihedral group D_k of symmetries of a regular k-gon.*
- *H is the tetrahedral group T (which has order 12)*
- *H is the octahedral group O (which has order 24)*
- *H is the icohosedral group I (which has order 60).*

2. Central extensions

The *center* of $\tilde{G}$ is

$$Z(\tilde{G}) = \{z \in \tilde{G} \mid \text{ if } g \in \tilde{G} \text{ then } zg = gz\}.$$

A *central extension* is a short exact sequence

$$1 \to C \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1.$$

such that $i(C) \subseteq Z(\tilde{G})$. Topic 2 of Tutorial 1 walks us through how to (re)construct $\tilde{G}$ after choosing a 2-cocycle $\beta \colon G \times G \to C$. WHAT IS THE PRECISE DEFINITION OF A 2-cocyle?

Example 2. The following is a central extension:

$$1 \to \{1, -1\} \xrightarrow{i} \{1, -1, i, -i\} \xrightarrow{p} \{1, -1\} \to 1$$

where

$$i(1) = 1, i(-1) = -1,$$
$$p(1) = 1,\ p(-1) = 1, p(i) = -1,\ p(-i) = -1.$$

Choose $\tilde{1} = -1 \in p^{-1}(1)$ and $\tilde{-1} = i \in p^{-1}(-1)$. Then, following Question 1, Topic 2, define

$$\beta(1,1) = -1,\ \beta(1,-1) = -1,\ \beta(-1,1) = -1,\ \beta(-1,-1) = 1.$$

Let

$$\tilde{G}_\beta = \{1, -1\} \times \{1, -1\}$$

where $\times$ means the Cartesian product (and *not* the direct product). Define a multiplication $\star \colon \tilde{G}_\beta \times \tilde{G}_\beta \to \tilde{G}_\beta$ by

$$(a, g) \star (b, h) = (ab\beta(g, h), gh).$$

Computing the multiplication table of $\tilde{G}_\beta$:

$\star$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,1)$	$(-1,1)$	$(1,-1)$	$(1,1)$	$(-1,-1)$
$(1,-1)$	$(-1,-1)$	$(1,1)$	$(1,-1)$	$(-1,1)$
$(-1,1)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(-1,-1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$	$(1,1)$

where the (i,j) entry is row $i \star$ column j. Using this table, one can check that $(-1,1)$ is the identity for $\tilde{G}_\beta$, $(1,1)$ has order 2, $(-1,-1)$ and $(1,-1)$ both have order 4. This suggests that the map $\psi \colon \tilde{G}_\beta \to \{1, -1, i, -i\}$ given by

$$\psi(-1,1) = 1, \psi(1,1) = -1, \psi(-1,-1) = i, \psi(1,-1) = -i$$

is an isomorphism of groups, but to prove this, we need to check that they have the same multiplication tables.

Example 3. (The Hopf Fibration). See §9.4 of Artin.

The 2-dimensional *special unitary group* is

$$\begin{aligned} SU(2) &= \left\{ A \in SL_2(\mathbb{C}) \mid A\overline{A}^t = 1 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \middle| a, b \in \mathbb{C}, a\overline{a} + b\overline{b} = 1 \right\}. \end{aligned}$$

The map

$$\begin{array}{ccc} SU_2 & \longrightarrow & S^3 \\ \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix} & \longmapsto & (x_0, x_1, x_2, x_3), \end{array}$$

is a bijection. Note that

$$\begin{array}{rcl} 1 & \longmapsto & (1,0,0,0) \\ i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} & \longmapsto & (0,1,0,0) \\ j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \longmapsto & (0,0,1,0) \\ i = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} & \longmapsto & (0,0,0,1). \end{array}$$

The *equator* of SU_2 is

$$E = \{P \in SU_2 \mid \mathrm{trace}(P) = 0\}.$$

In the 3-sphere S^3, the equator corresponds to

$$E = \{(0, x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Given $P \in SU_2$, the conjugation action by P on E, denoted γ_P, rotates the sphere $E = S^2$ (see Artin). This gives us a surjective homomorphism

$$\begin{array}{rcl} p\colon SU(2) & \longrightarrow & SO(3) \\ P & \longmapsto & \gamma_P \end{array}$$

and the following is a central extension:

$$1 \to \{1, -1\} \xrightarrow{i} SU(2)(\cong \mathrm{Spin}(3)) \xrightarrow{p} SO(3) \to 1.$$

This should help us understand Topic 2, Question 5, since the groups mentioned are finite subgroups of $SU(2)$ and $SO(3)$.

3. The classification of finitely generated modules over PIDs, and its corollaries

Theorem 3.1 (Theorem 5, Section 12.1, Dummit-Foot)**.** *(Fundamental Theorem, Existence: Invariant Factor Form) Let R be a PID and let M be a finitely generated R-module. Then*

(1) There exists $r \in \mathbb{Z}_{\geq 0}$ and $a_1, a_2, \dots, a_m \in R$ such that $a_1, a_2, \dots, a_m$ are not units in R and $a_1 | a_2 | \dots | a_m$ such that

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m).$$

(2) M is torsion-free if and only if M is free.

(3)

$$Tor(M) \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m).$$

In particular, M is a torsion module if and only if $r = 0$ and in this case the annihilator of M is the ideal (a_m).

3.1. The Jordan normal Form.

4. Miscellaneous

4.1. Generators and Relations.

From Arun Ram's website and §6.3 of Dummit and Foote.

Let S be a set (sometimes called the set of *letters*). A *free group* on S is a group $F(S)$ such that if G is a group and $\psi\colon S \to G$ is a function then there exists a unique homomorphism $\Phi\colon F(S) \to G$ such that $\Phi \circ i = \psi$.

$$\begin{array}{ccc} S & \overset{i}{\hookrightarrow} & F(S) \\ & \searrow^{\psi} & \downarrow{\scriptstyle\Phi} \\ & & G \end{array}$$

A *word* on S is a sequence

$$(s_1, s_2, \ldots) \text{ where } s_i \in S \cup S^{-1} \cup \{1\} \text{ and } s_i = 1 \text{ for all } i \text{ sufficiently large.}$$

A word $(s_1, s_2, \ldots)$ is *reduced* if

(1) $s_{i+1} \neq s_i^{-1}$ for all i with $s_i \neq 1$, and
(2) if $s_k = 1$ for some k, then $s_i = 1$ for all $i \geq k$.

Let

$$F(S) \text{ be the set of reduced words on } S.$$

Proposition 4.1. *$F(S)$ is the unique (up to isomorphism) free group on S.*

Let G be a group. A *presentation* of G is an exact sequence

$$R \to H \to G \to 1$$

where R and H are free groups. A group G is *presented* by generators $g_1, g_2, \ldots, g_n$ and relations $r_1, r_2, \ldots, r_m$ if

$$F\{y_1, y_2, \ldots, y_m\} \xrightarrow{\psi} F\{x_1, x_2, \ldots, x_n\} \xrightarrow{p} G \to 1.$$

is an exact sequence, where

$$\psi(y_j) = r_j, \quad p(x_i) = g_i.$$

Write

$$G = \langle g_1, g_2, \dots, g_n \mid r_1 = r_2 = \dots = r_m = 1 \rangle$$

where $r_i = r_i(g_1, g_2, \dots, g_n)$ are words in $g_1, g_2, \dots, g_n$. This means that

$$G = F(S)/N(r_1, r_2, \dots, r_m)$$

where $N(r_1, r_2, \dots, r_m)$ is the minimal normal subgroup containing $r_1, r_2, \dots, r_m$.

Proposition 4.2. *Let $\hat{f}\colon \{g_1, g_2, \dots, g_n\} \to H$ be a function. If the relations $r_1 = r_2 = \dots = r_k = 1$ are also satisfied by $\hat{f}(g_1), \hat{f}(g_2), \dots \hat{f}(g_n)$ then $\hat{f}$ uniquely extends to a homomorphism $f\colon G \to H$.*

Proof. Let

$$F\{y_1, y_2, \dots, y_m\} \xrightarrow{\psi} F\{x_1, x_2, \dots, x_n\} \xrightarrow{\pi} G \to 1.$$

be a presentation of G, where

$$\psi(y_j) = r_j, \quad \pi(x_i) = g_i.$$

Define $\pi'\colon F\{x_1, x_2, \dots, x_n\} \to H$ by defining

$$\pi'(x_i) = \hat{f}(g_i)$$

and uniquely extending to the rest of $F\{x_1, x_2, \dots, x_n\}$ by the universal property of free groups. Then $\ker\pi \subseteq \ker\pi'$ (HUH??? I THINK THIS IS WHAT 'the relations $r_1 = r_2 = \dots = r_k = 1$ are also satisfied by $\hat{f}(g_1), \hat{f}(g_2), \dots \hat{f}(g_n)$' MEANS) so that the map

$$\begin{aligned} f\colon G \cong F\{x_1, x_2, \dots, x_n\}/\ker\pi &\longrightarrow H \cong F\{x_1, x_2, \dots, x_n\}/\ker\pi' \\ g\ker\pi &\longmapsto g\ker\pi' \end{aligned}$$

is a well defined group homomorphism. □